The debiased Keyl's algorithm

a new, unbiased estimator for full-state tomography

Angelos Pelecanos   Jack Spilecki   John Wright

Berkeley
UNIVERSITY OF CALIFORNIA

Goal: Construct unbiased estimator for entangled quantum state tomography.

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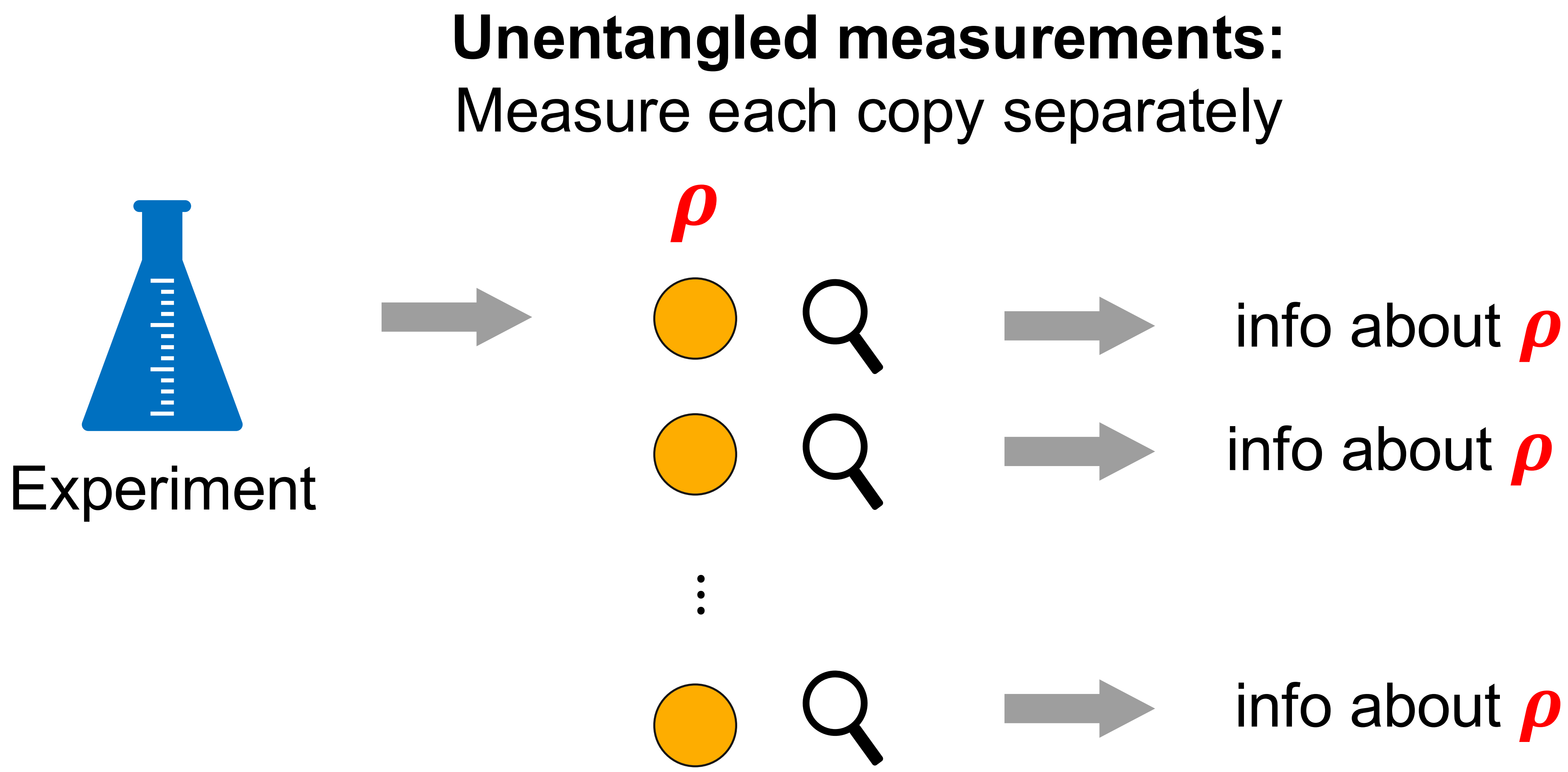
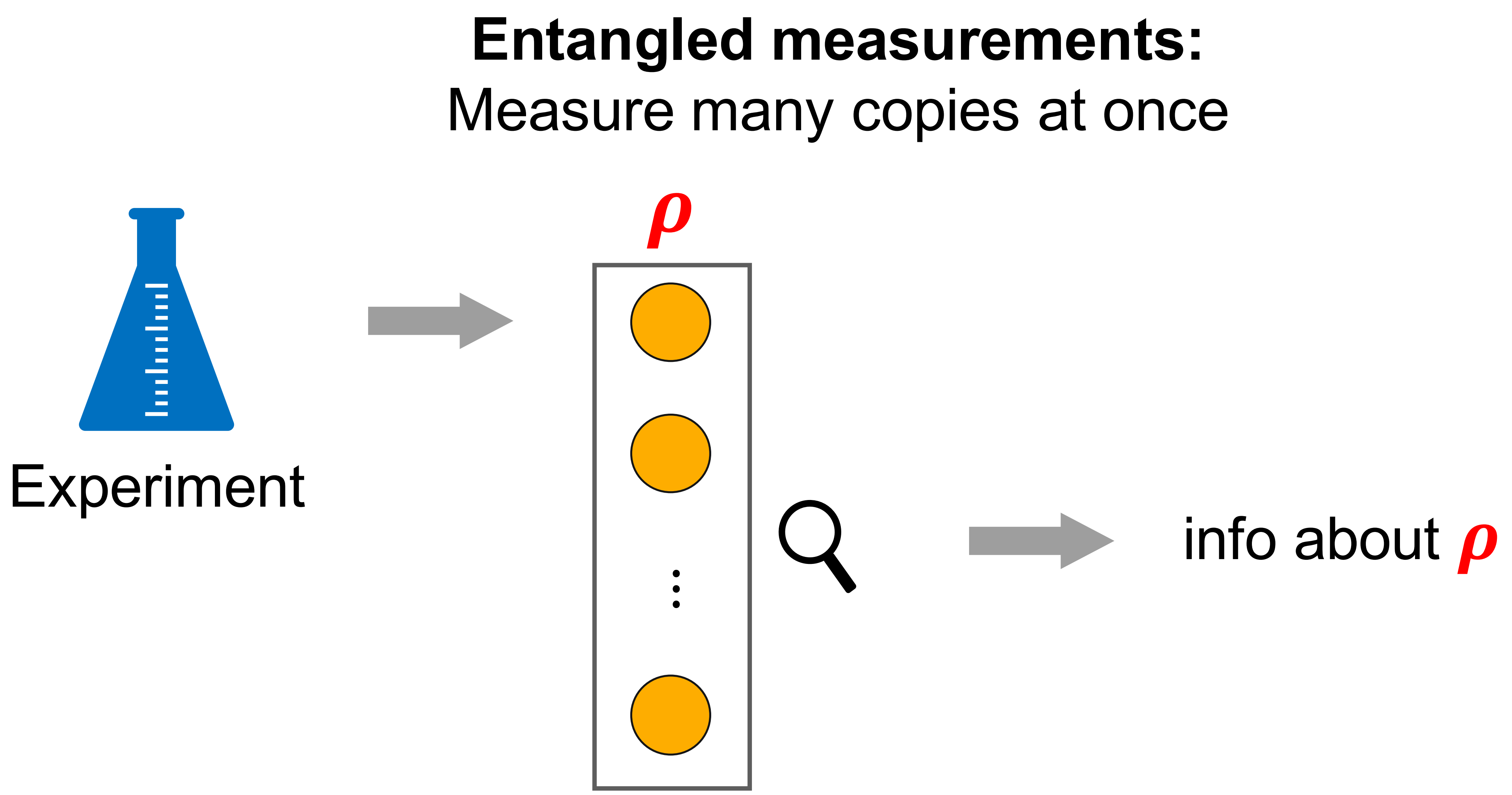
Quantum State Tomography



Given: n copies of a quantum state ρ.

Goal: Learn something about ρ.

Quantum State Tomography



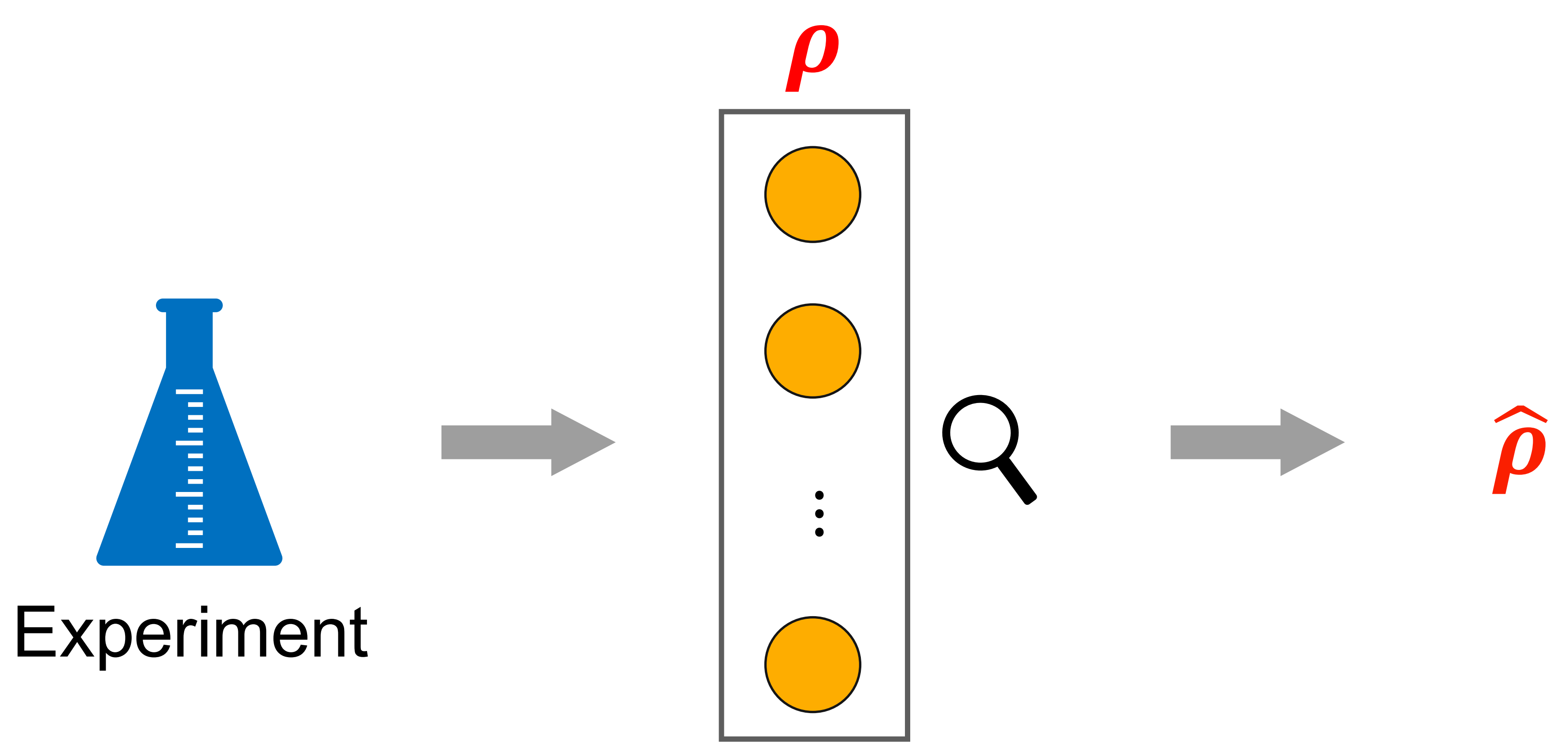
Given: n copies of a quantum state ρ.

Goal: Learn something about ρ.

Quantum state tomography: Learn the entire state ρ.

More tomography problems later on.

Quantum State Tomography



Given: n copies of a rank state ρ.

Goal: Output $\hat{\rho}$ such that $D(\hat{\rho}, \rho) \leq \varepsilon$ with probability 99%.

D will be **trace distance** D_{tr} or **infidelity** $1 - F$.

Samples: Find the minimum number of copies n.

Prior work on state tomography

Given: $\boldsymbol{n}$ copies of a rank state $\boldsymbol{\rho}$.

Goal: Output $\widehat{\boldsymbol{\rho}}$ such that $\boldsymbol{D(\widehat{\rho}, \rho) \leq \varepsilon}$ with probability $\mathbf{99\%}$.

Three (almost) sample-optimal algorithms - analyzed in **[HHJWY16,OW16]**.

All three: representation-theoretic, hard to analyze.

One is **Keyl's algorithm** **[Keyl06]**, further studied in **[OW16]**.

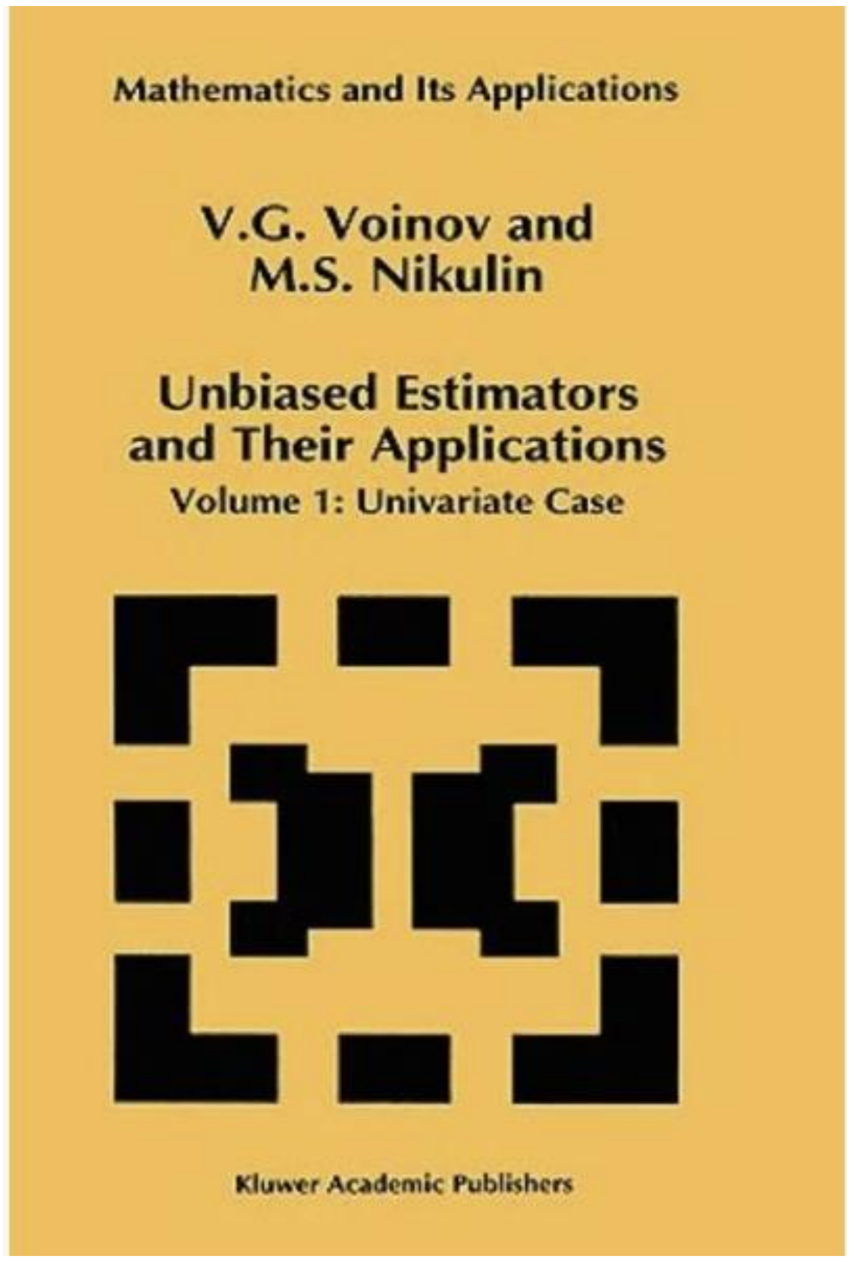
Goal: Construct unbiased estimator for entangled quantum state tomography.

Goal: Construct **unbiased estimator** for entangled quantum state tomography.

Unbiased estimators

Definition: An estimator is unbiased if $\boldsymbol{E[\hat{\rho}] = \rho}$.

Unbiased estimators have many desirable features.



> "Unbiasedness is one of the most important properties of statistical estimators."
>
> Voinov and Nikulin

Collect ≈ 1000 unbiased estimators for different statistical parameters.

Theoretical motivation: Construct unbiased estimators for fundamental problems.

Also: Unbiased estimators give simple algorithms for quantum state tomography tasks.

Unbiased estimators for state tomography

Definition: An estimator is unbiased if $\boldsymbol{E}[\widehat{\boldsymbol{\rho}}] = \boldsymbol{\rho}$.

The known sample-optimal algorithms are very biased.

Biases not well understood.

Examples of unbiased estimators in state tomography.

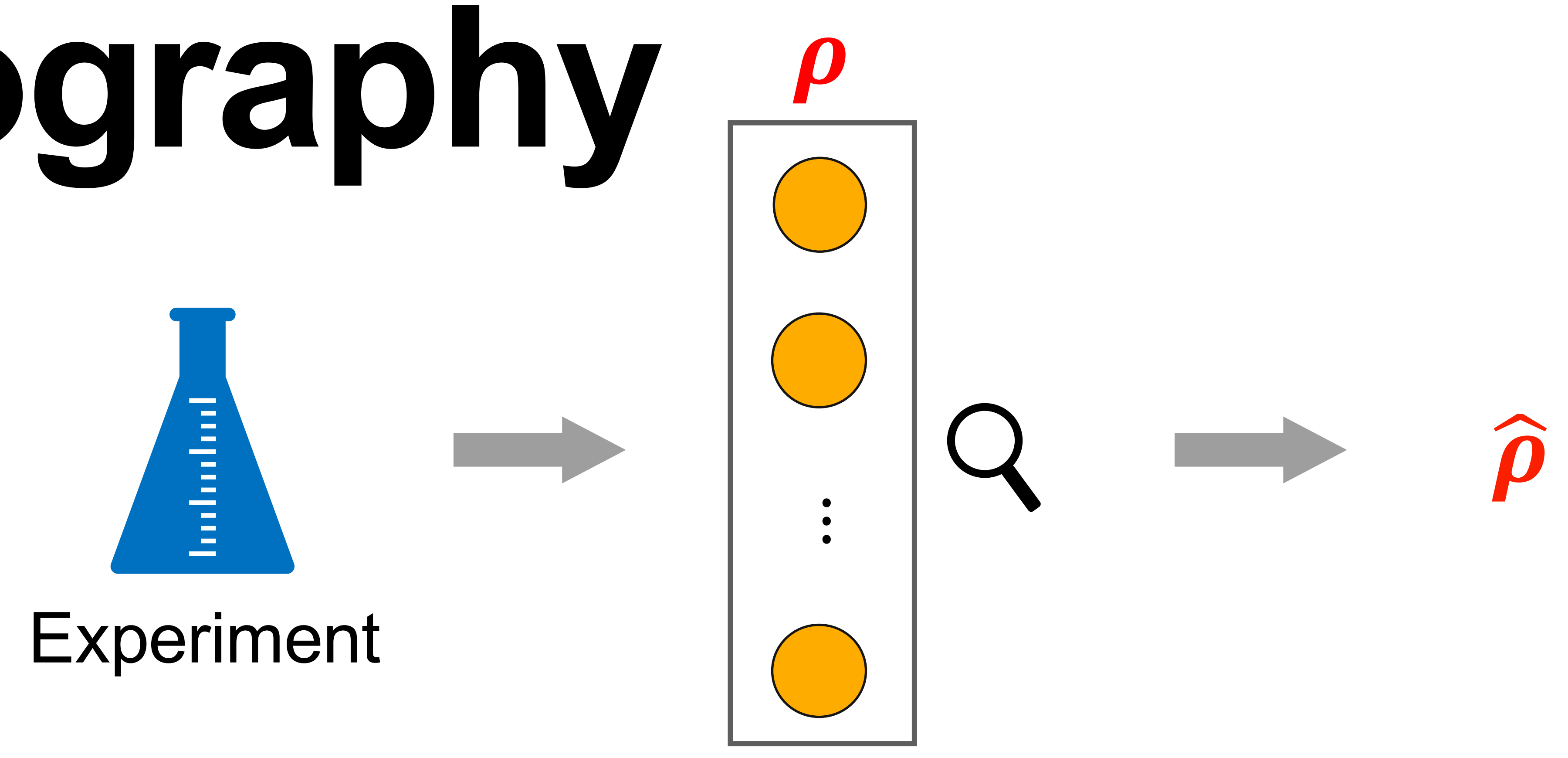
1. Unentangled tomography algorithms.

Cannot be sample-optimal [CHLL23]

2. Pure state tomography algorithm of [GPS22].

Goal: Construct unbiased estimator for entangled quantum state tomography.

Applications of State Tomography



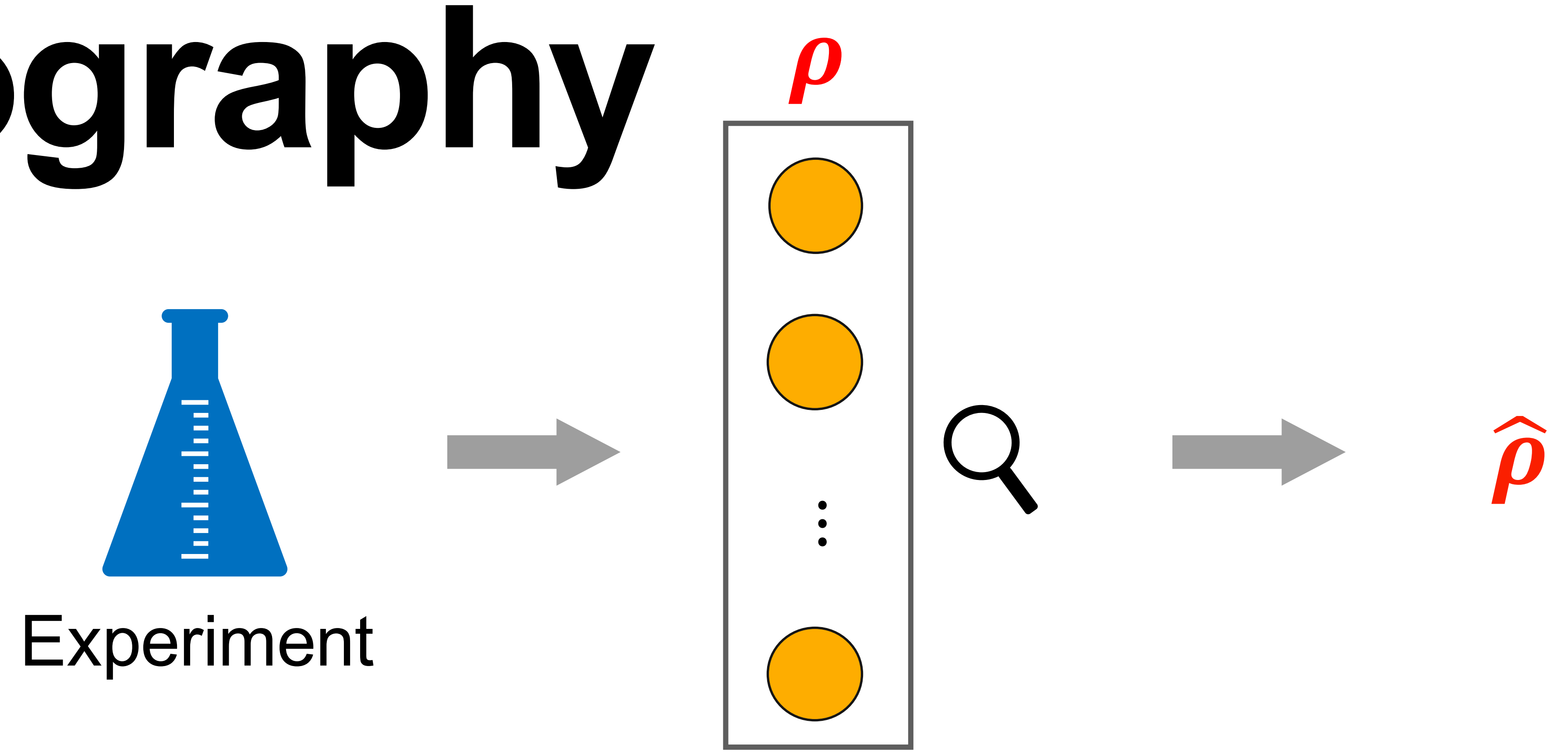
Goal: Learn something about ρ.

Quantum state tomography: Learn the entire state ρ.

Recent works by Chen, Li, Liu [CLL24a,CLL24b] studied fundamental applications.

Bottleneck: no **unbiased estimators** for entangled tomography.

Applications of State Tomography



Goal: Learn something about $\boldsymbol{\rho}$.

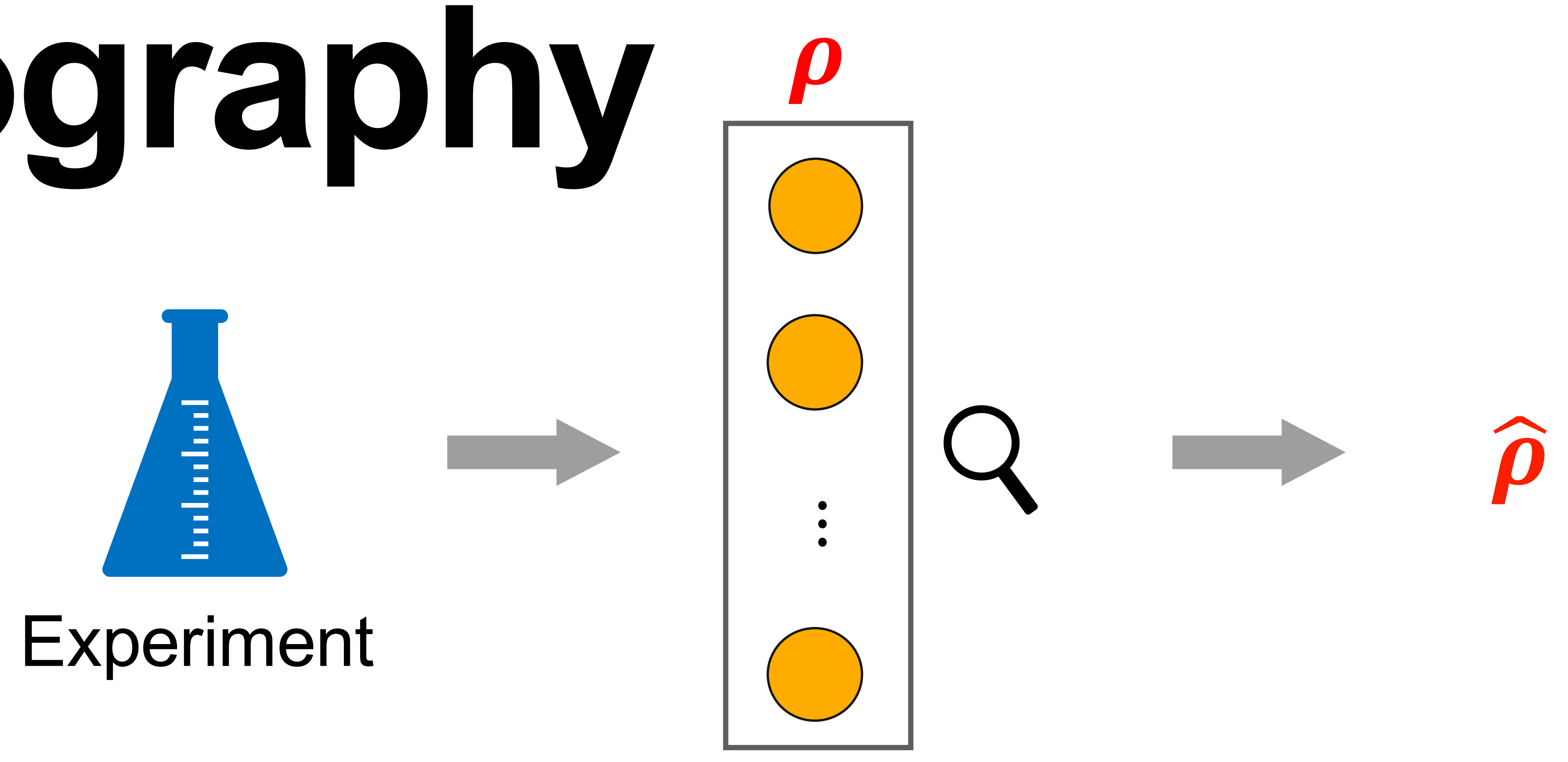
Quantum state tomography: Learn the entire state $\boldsymbol{\rho}$.

$\boldsymbol{k}$-entangled tomography: Estimate $\boldsymbol{\rho}$ to distance $\boldsymbol{\varepsilon}$ by measuring $\boldsymbol{k}$ copies at a time.

Near-term devices may only implement smaller entangled measurements.

Interpolates between unentangled ($\boldsymbol{k = 1}$) and entangled ($\boldsymbol{k = n}$) measurements.

Applications of State Tomography



Goal: Learn something about $\boldsymbol{\rho}$.

Quantum state tomography: Learn the entire state $\boldsymbol{\rho}$.

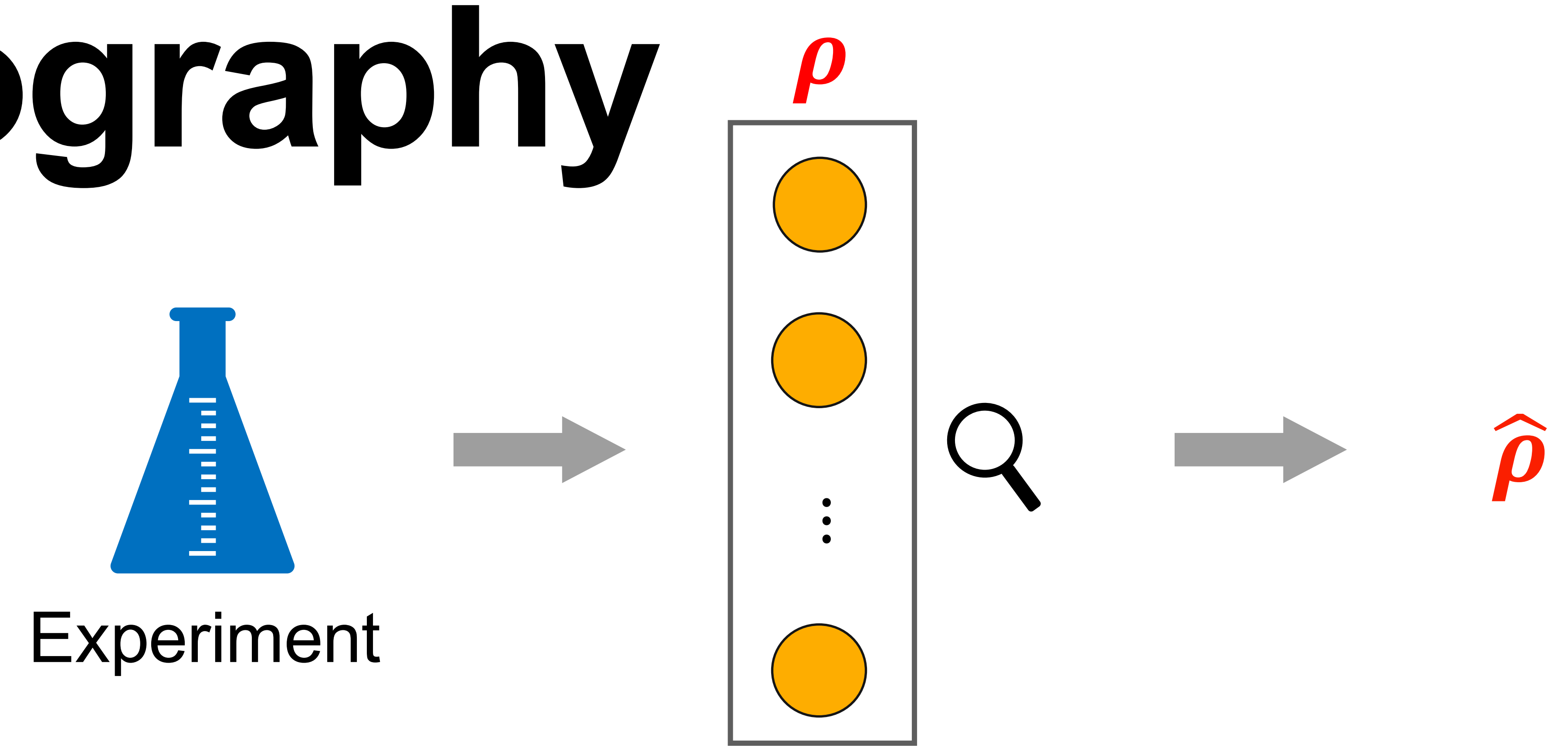
$\boldsymbol{k}$**-entangled tomography:** Estimate $\boldsymbol{\rho}$ to distance $\boldsymbol{\varepsilon}$ by measuring $\boldsymbol{k}$ copies at a time.

Shadow tomography: Observables $\boldsymbol{O_1, \ldots, O_m}$,estimate $\mathrm{tr}(\boldsymbol{O_1} \cdot \boldsymbol{\rho}), \ldots, \mathrm{tr}(\boldsymbol{O_m} \cdot \boldsymbol{\rho})$ to error $\boldsymbol{\varepsilon}$.

Many applications to learning, testing among others.

Often much more efficient than full-state tomography.

Applications of State Tomography



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We also identify a third application:

Quantum metrology: Given parametrized state $\boldsymbol{\rho_\theta}$,estimate $\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{m}}$.

Unbiasedness gives simple algorithms

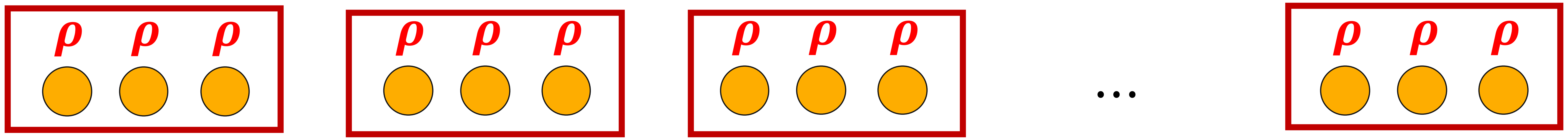
Definition: An estimator is unbiased if $E[\hat{\rho}] = \rho$.

Application 1: k-entangled tomography.

Estimate ρ to distance ε by measuring k copies at a time.

1. Divide the n copies of ρ into $m = n/k$ groups.

k copies



$\hat{\rho}_1$ $\hat{\rho}_2$ $\hat{\rho}_3$ $\hat{\rho}_m$

2. Run unbiased tomography on each group.

3. Output $\hat{\rho} = \frac{1}{m} \cdot (\hat{\rho}_1 + \cdots + \hat{\rho}_m)$.

also unbiased

average decreases variance

Main results

Introduce the first unbiased estimator for entangled tomography.

The **debiased Keyl's** algorithm: modification of Keyl's to remove bias.

Theorem. Let $\widehat{\boldsymbol{\rho}}$ be the output of debiased Keyl's. Then $\boldsymbol{E}[\widehat{\boldsymbol{\rho}}] = \boldsymbol{\rho}$.

Quality of an unbiased estimator is given by its "variance".

Theorem. The second moment of $\widehat{\boldsymbol{\rho}}$ satisfies:

$$\boldsymbol{E}[\widehat{\boldsymbol{\rho}} \otimes \widehat{\boldsymbol{\rho}}] = \frac{\boldsymbol{n}-\boldsymbol{1}}{\boldsymbol{n}} \cdot \boldsymbol{\rho} \otimes \boldsymbol{\rho} + \frac{\boldsymbol{1}}{\boldsymbol{n}} \cdot (\boldsymbol{\rho} \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \boldsymbol{\rho}) \cdot \mathrm{SWAP} + \frac{\boldsymbol{E}[\boldsymbol{\ell}(\boldsymbol{\lambda})]}{\boldsymbol{n}^{2}} \cdot \mathrm{SWAP} - \mathrm{Lower}_{\boldsymbol{\rho}}$$

What is $\ell(\lambda)$?
Coming up soon…

Debiased Keyl's algorithm

Unbiased: $E[\hat{\rho}] = \rho$.

Variance:

$$E[\hat{\rho} \otimes \hat{\rho}] = \frac{n-1}{n} \cdot \rho \otimes \rho + \frac{1}{n} \cdot (\rho \otimes I + I \otimes \rho) \cdot \mathrm{SWAP} + \frac{E[\ell(\lambda)]}{n^2} \cdot \mathrm{SWAP} - \mathrm{Lower}_{\rho}$$

May seem intimidating at first.

The debiased Keyl's algorithm: a new unbiased estimator for full state tomography 1 / 128

This expression was discovered with a lot of:

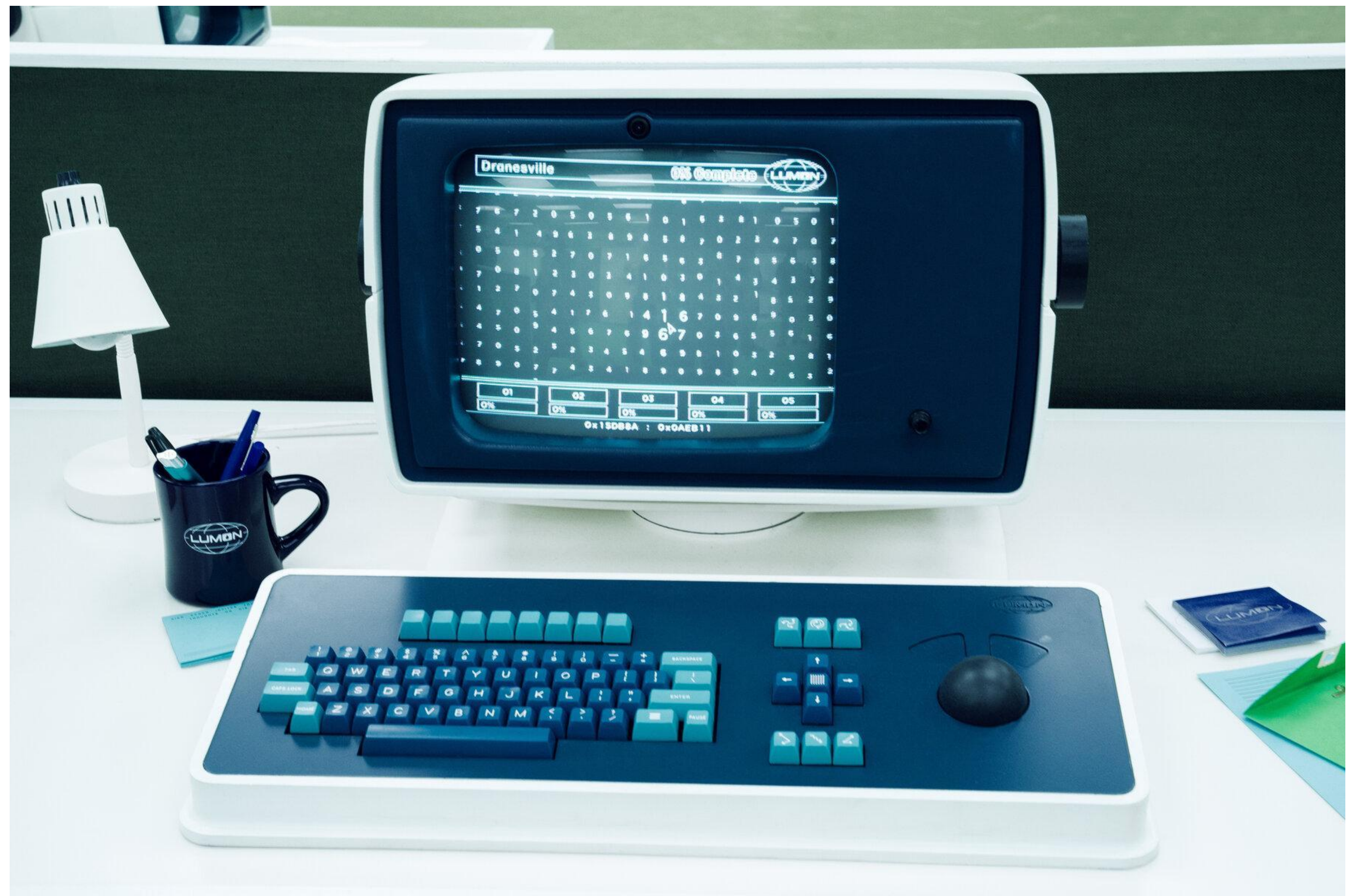


Image credit: https://movies.stackexchange.com/questions/119335/are-the-computers-in-severance-real

Debiased Keyl's algorithm

Unbiased: $\boldsymbol{E[\hat{\rho}] = \rho}$.

Variance:

$$\boldsymbol{E[\hat{\rho} \otimes \hat{\rho}] = \frac{n-1}{n} \cdot \rho \otimes \rho + \frac{1}{n} \cdot (\rho \otimes I + I \otimes \rho)} \cdot \mathrm{SWAP} + \boldsymbol{\frac{E[\ell(\lambda)]}{n^2}} \cdot \mathrm{SWAP} - \mathrm{Lower}_{\boldsymbol{\rho}}$$

May seem intimidating at first.

Actually very easy to use!

Involves nice matrices.

Debiased Keyl's algorithm

Unbiased: $E[\hat{\rho}] = \rho$.

Variance:

$$E[\hat{\rho} \otimes \hat{\rho}] = \boxed{\frac{n-1}{n} \cdot \rho \otimes \rho} + \frac{1}{n} \cdot (\rho \otimes I + I \otimes \rho) \cdot \mathrm{SWAP} + \frac{E[\ell(\lambda)]}{n^2} \cdot \mathrm{SWAP} - \mathrm{Lower}_{\rho}$$

Main term: ideally $E[\hat{\rho} \otimes \hat{\rho}] = \rho \otimes \rho$

Debiased Keyl's algorithm

Unbiased: $E[\hat{\rho}] = \rho$.

Variance:

$$E[\hat{\rho} \otimes \hat{\rho}] = \frac{n-1}{n} \cdot \rho \otimes \rho + \boxed{\frac{1}{n} \cdot (\rho \otimes I + I \otimes \rho) \cdot \mathrm{SWAP}} + \frac{E[\ell(\lambda)]}{n^2} \cdot \mathrm{SWAP} - \mathrm{Lower}_\rho$$

Main error term:
Scales with $1/n$ like variance.

Debiased Keyl's algorithm

Unbiased: $E[\hat{\rho}] = \rho$.

Variance:

$$E[\hat{\rho} \otimes \hat{\rho}] = \frac{n-1}{n} \cdot \rho \otimes \rho + \frac{1}{n} \cdot (\rho \otimes I + I \otimes \rho) \cdot \mathrm{SWAP} + \boxed{\frac{E[\ell(\lambda)]}{n^2} \cdot \mathrm{SWAP}} - \mathrm{Lower}_{\rho}$$

Second-order error term:
Matters only for small n.

Debiased Keyl's algorithm

Unbiased: $E[\hat{\rho}] = \rho$.

Variance:

$$E[\hat{\rho} \otimes \hat{\rho}] = \frac{n-1}{n} \cdot \rho \otimes \rho + \frac{1}{n} \cdot (\rho \otimes I + I \otimes \rho) \cdot \mathrm{SWAP} + \frac{E[\ell(\lambda)]}{n^2} \cdot \mathrm{SWAP} - \boxed{\mathrm{Lower}_\rho}$$

Lower-order error term:
Vanishes for large d.
Negative for our applications.

Debiased Keyl's algorithm

Unbiased: $\boldsymbol{E}[\widehat{\boldsymbol{\rho}}] = \boldsymbol{\rho}$.

Variance:

$$\boldsymbol{E}[\widehat{\boldsymbol{\rho}} \otimes \widehat{\boldsymbol{\rho}}] = \frac{\boldsymbol{n}-\mathbf{1}}{\boldsymbol{n}} \cdot \boldsymbol{\rho} \otimes \boldsymbol{\rho} + \frac{\mathbf{1}}{\boldsymbol{n}} \cdot (\boldsymbol{\rho} \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \boldsymbol{\rho}) \cdot \mathrm{SWAP} + \frac{\boldsymbol{E}[\ell(\boldsymbol{\lambda})]}{\boldsymbol{n}^{2}} \cdot \mathrm{SWAP} - \mathrm{Lower}_{\boldsymbol{\rho}}$$

Very clean and easy to use.

All learning applications follow from the two properties above.

Debiased Keyl's is sample-optimal

Goal: Estimate $\boldsymbol{\rho}$ to distance $\boldsymbol{\varepsilon}$ using entangled measurements.

Theorem. Can estimate rank-$\boldsymbol{r}$ state $\boldsymbol{\rho} \in \mathbb{C}^{\boldsymbol{d \times d}}$ using

$\boldsymbol{n} = \boldsymbol{O}\left(\frac{\boldsymbol{rd}}{\boldsymbol{\varepsilon^2}}\right)$ samples to $\boldsymbol{\varepsilon}$ trace distance

$\boldsymbol{n} = \boldsymbol{O}\left(\frac{\boldsymbol{rd}}{\boldsymbol{\varepsilon}}\right)$ samples to $\boldsymbol{\varepsilon}$ infidelity

Matches known lower bounds **[Yuen23, SSW25]**.

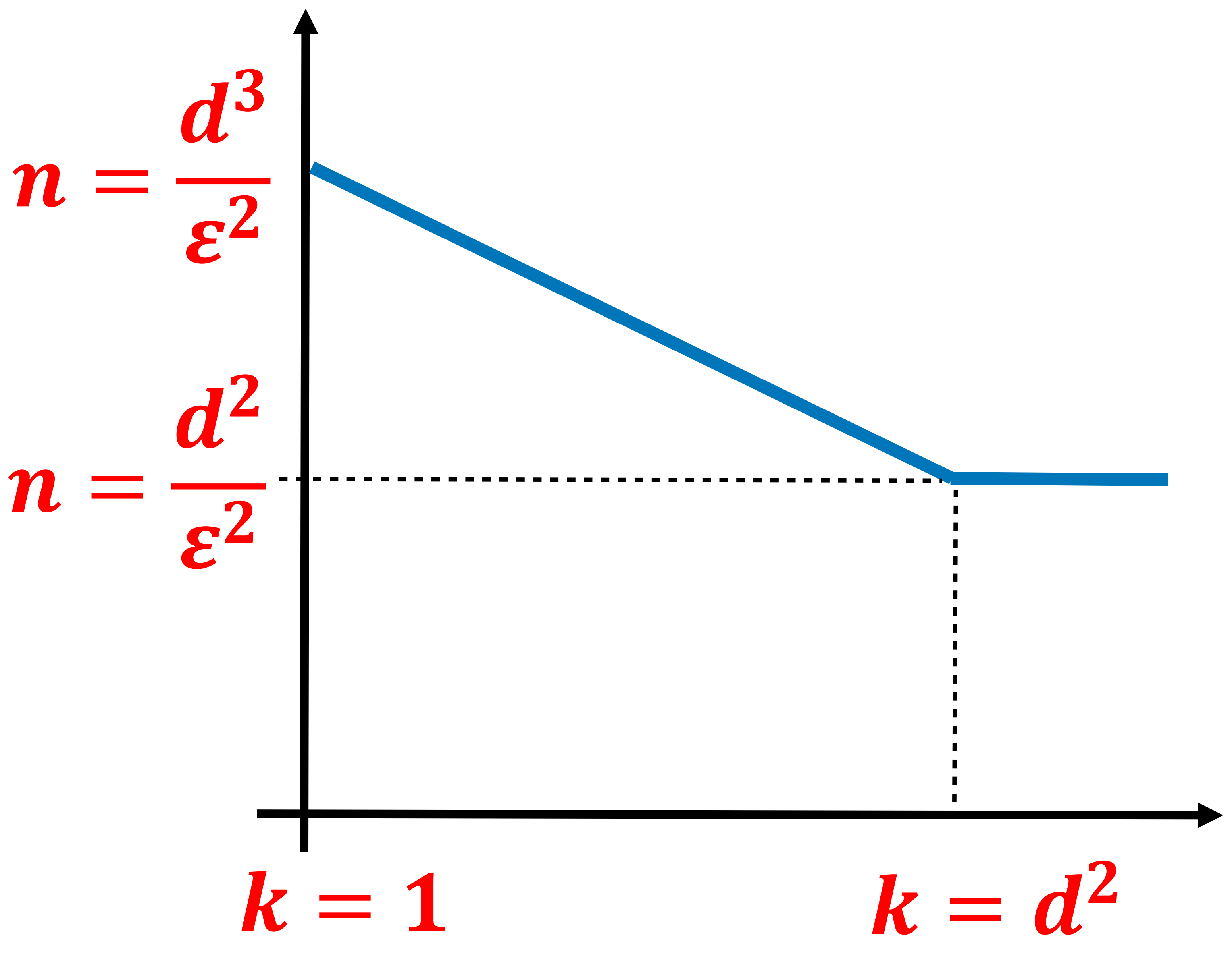
Improves upon prior infidelity bounds.

Application 1: k-entangled tomography

Goal: Estimate ρ to distance ε by measuring k copies at a time.

Theorem. Samples $n = O\left(\frac{d^3}{\sqrt{k}\varepsilon^2} + \frac{d^2}{\varepsilon^2}\right)$ suffice to trace distance ε.

Matches bounds for unentangled and entangled measurements.



Previous work: $n = \tilde{O}\left(\frac{d^3}{\sqrt{k}\varepsilon^2}\right)$ only for "small" k [CLL24a].

$n = \Omega\left(\frac{d^3}{\sqrt{k}\varepsilon^2}\right)$ only for "small" k [CLL24a].

Application 2: Shadow tomography

Goal: Observables $\boldsymbol{O_1, \dots, O_m}$,estimate $\mathrm{tr}(\boldsymbol{O_1} \cdot \boldsymbol{\rho}), \dots, \mathrm{tr}(\boldsymbol{O_m} \cdot \boldsymbol{\rho})$ to error $\boldsymbol{\varepsilon}$.

Theorem. Samples $\boldsymbol{n} = \boldsymbol{O}(\log(\boldsymbol{m})/\boldsymbol{\varepsilon}^2)$ suffice when $\boldsymbol{\varepsilon} = \boldsymbol{O}(\mathbf{1}/\boldsymbol{d})$.

- Sample-optimal in high-precision regime.

Previous work: Samples $\boldsymbol{n} = \boldsymbol{O}(\log(\boldsymbol{m})/\boldsymbol{\varepsilon}^2)$ suffice when $\boldsymbol{\varepsilon} = \boldsymbol{O}(\mathbf{1}/\boldsymbol{d}^{\mathbf{12}})$]CLL24b].

Classical shadows: Obtain a more general bound.

- Improves upon prior work [GLM24].
- Disproves a conjecture by [CLL24b].

Application 3: Multi-parameter quantum metrology

Goal: Given parametrized state $\boldsymbol{\rho_\theta}$,estimate $\boldsymbol{\theta} \in \mathbb{R}^{\boldsymbol{m}}$.

Well-known bound called the **quantum Cramér-Rao bound**.

We attain twice this bound, which turns out to be optimal.

The bound we obtain has been stated before, but without proof.

Interesting: our techniques are very different from prior work.

The algorithm

Keyl's algorithm

Our starting point is Keyl's algorithm.

Input: n copies of mixed state $\rho \in \mathbb{C}^{d \times d}$.

Let $\rho = U \cdot \mathrm{diag}(\alpha) \cdot U^{-1}$ with sorted eigenvalues $\alpha_1 \geq \cdots \geq \alpha_d$.

Two steps:

1. **Estimate the spectrum** α.

   Perform weak Schur sampling. Outcome is Young diagram λ.

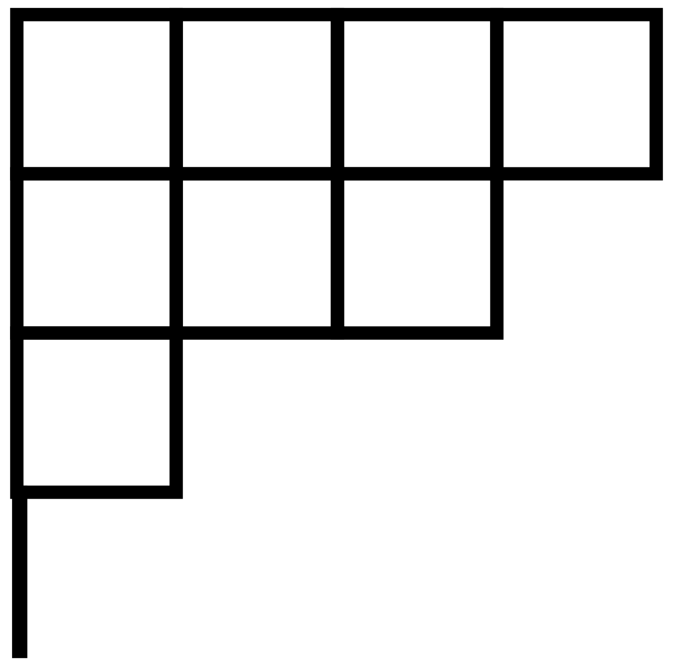
Young diagrams:

consist of n boxes $\quad n = 8$

in d rows (rows possibly empty) $\quad d = 4$

rows (weakly) decreasing

$\ell(\lambda)$ is # non-empty rows



$\lambda = (4, 3, 1, 0)$

$\lambda_1 = 4$

$\lambda_2 = 3$

$\lambda_3 = 1$

$\lambda_4 = 0$

Keyl's algorithm

Our starting point is Keyl's algorithm.

Input: $\boldsymbol{n}$ copies of mixed state $\boldsymbol{\rho} \in \mathbb{C}^{\boldsymbol{d} \times \boldsymbol{d}}$.

Let $\boldsymbol{\rho} = \boldsymbol{U} \cdot \mathrm{diag}(\boldsymbol{\alpha}) \cdot \boldsymbol{U}^{-1}$ with sorted eigenvalues $\boldsymbol{\alpha_1} \geq \cdots \geq \boldsymbol{\alpha_d}$.

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   Let $\boldsymbol{\lambda} = (\boldsymbol{\lambda_1}, \dots, \boldsymbol{\lambda_d})$. Spectrum estimate is $\widehat{\boldsymbol{\alpha}} = \boldsymbol{\lambda}/\boldsymbol{n} = (\boldsymbol{\lambda_1}/\boldsymbol{n}, \dots, \boldsymbol{\lambda_d}/\boldsymbol{n})$.

2. **Estimate the eigenvectors** $\boldsymbol{U}$.

   Further measurement to estimate eigenvectors $\widehat{\boldsymbol{U}}$.

Output: State estimate $\widehat{\boldsymbol{\rho}} = \widehat{\boldsymbol{U}} \cdot \mathrm{diag}(\widehat{\boldsymbol{\alpha}}) \cdot \widehat{\boldsymbol{U}}^{-1}$.

Debiased Keyl's algorithm

Our starting point is Keyl's algorithm.

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Output: State estimate $\widehat{\boldsymbol{\rho}} = \widehat{\boldsymbol{U}} \cdot \mathrm{diag}(\widehat{\boldsymbol{\alpha}}) \cdot \widehat{\boldsymbol{U}}^{-1}$.

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   Let $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_d)$.

   Spectrum estimate is $\widehat{\boldsymbol{\alpha}} = \boldsymbol{\lambda}^{\uparrow}/\boldsymbol{n} = \left(\boldsymbol{\lambda}_1^{\uparrow}/\boldsymbol{n}, \dots, \boldsymbol{\lambda}_d^{\uparrow}/\boldsymbol{n}\right)$.

2. **Estimate the eigenvectors** $\boldsymbol{U}$.

   Further measurement to estimate eigenvectors $\widehat{\boldsymbol{U}}$.

Output: State estimate $\widehat{\boldsymbol{\rho}} = \widehat{\boldsymbol{U}} \cdot \mathrm{diag}(\widehat{\boldsymbol{\alpha}}) \cdot \widehat{\boldsymbol{U}}^{-1}$.

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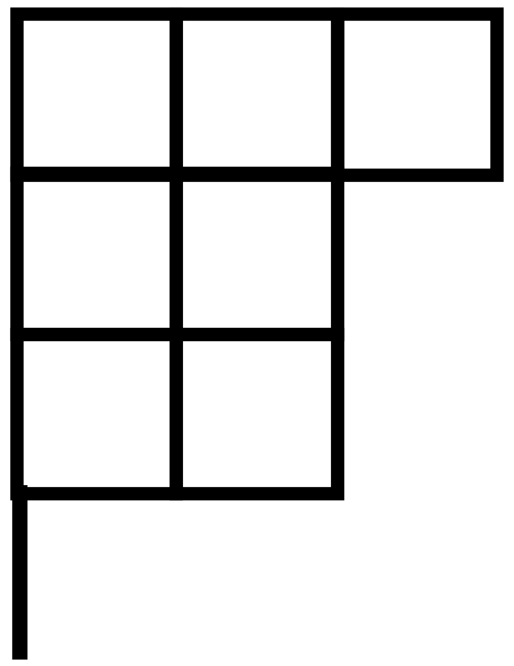
2. **Estimate the eigenvectors** $\boldsymbol{U}$.

   Further measurement to estimate eigenvectors $\widehat{\boldsymbol{U}}$.

Output: State estimate $\widehat{\boldsymbol{\rho}} = \widehat{\boldsymbol{U}} \cdot \mathrm{diag}(\widehat{\boldsymbol{\alpha}}) \cdot \widehat{\boldsymbol{U}}^{-1}$.

What is $\lambda^{\uparrow}$?

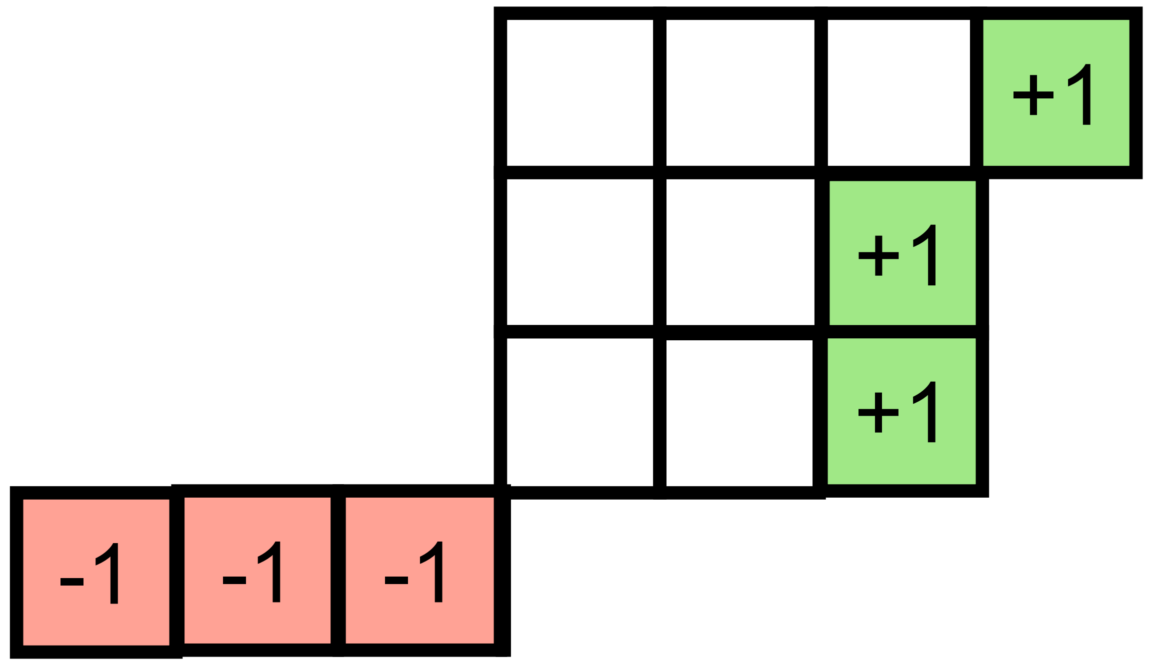
Start from Young diagram λ.



$\boldsymbol{\lambda = (3, 2, 2, 0)}$

Rule. Each row “donates” one box to each row that is strictly longer.

Last row donates to all other rows.



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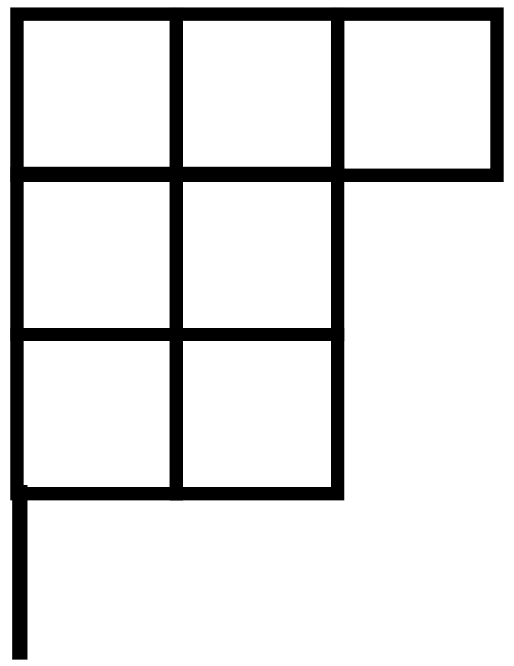
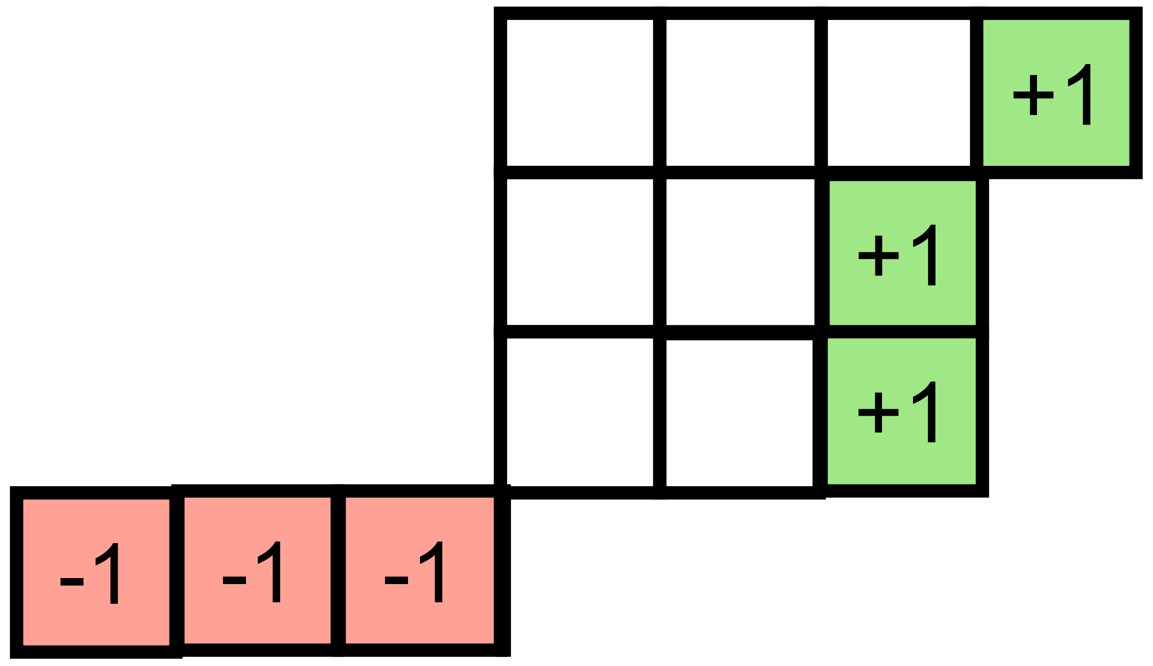


$\lambda = (3, 2, 2, 0)$

Rule. Each row “donates” one box to each row that is strictly longer.

Third row donates to first row.



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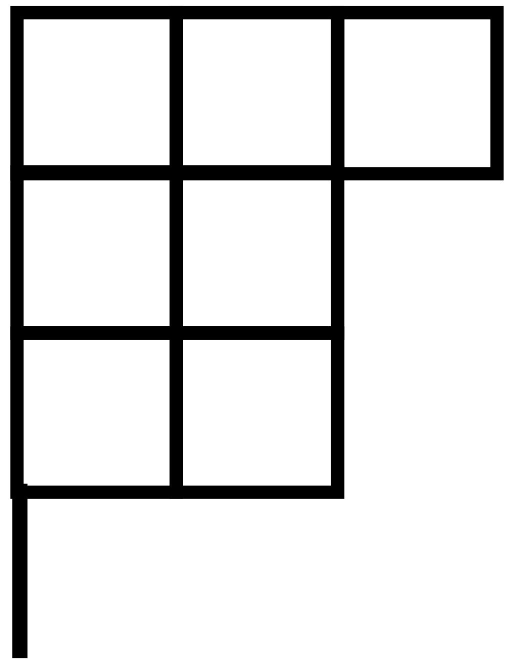

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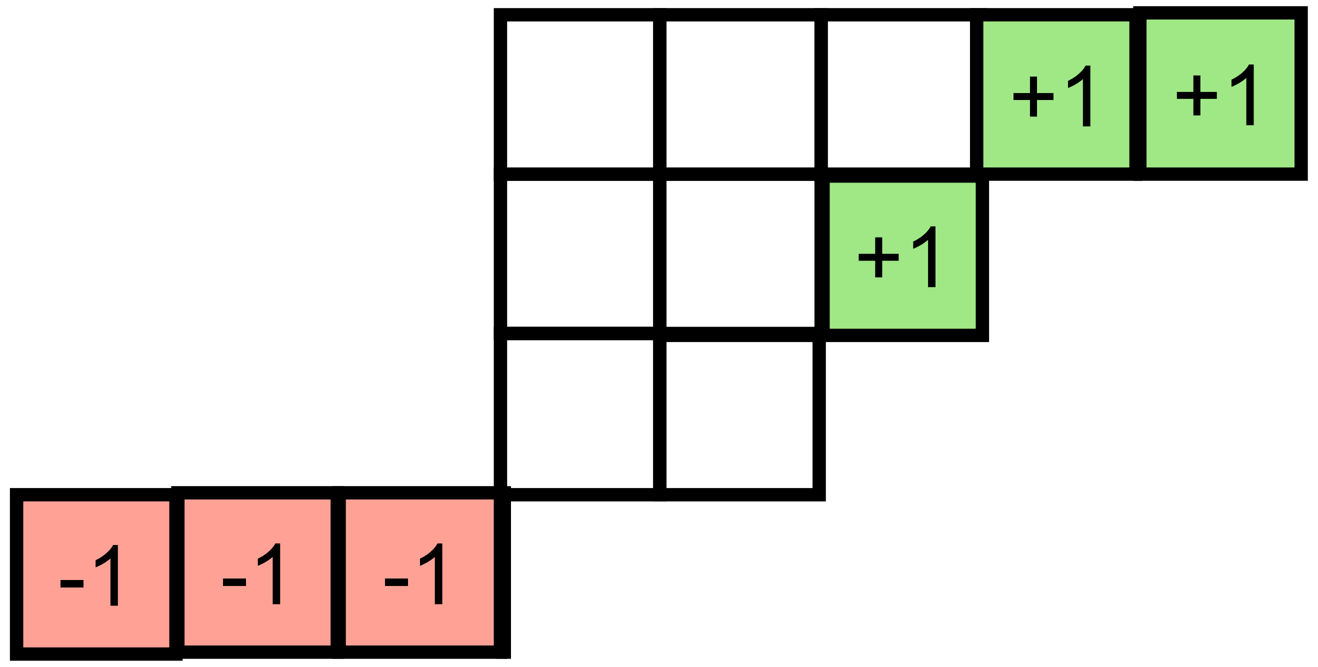

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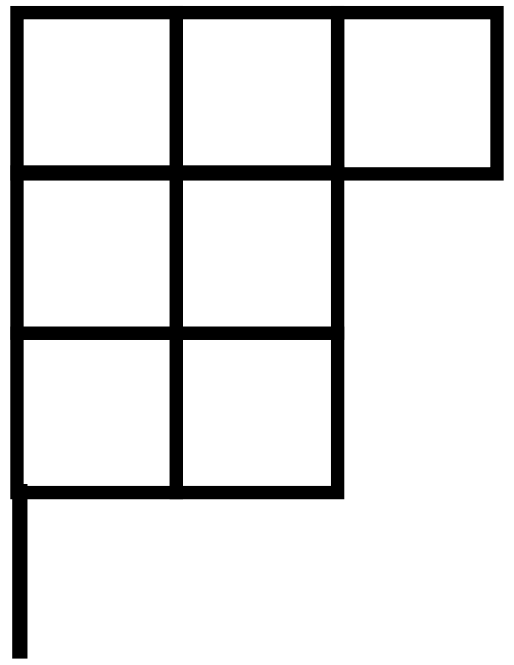
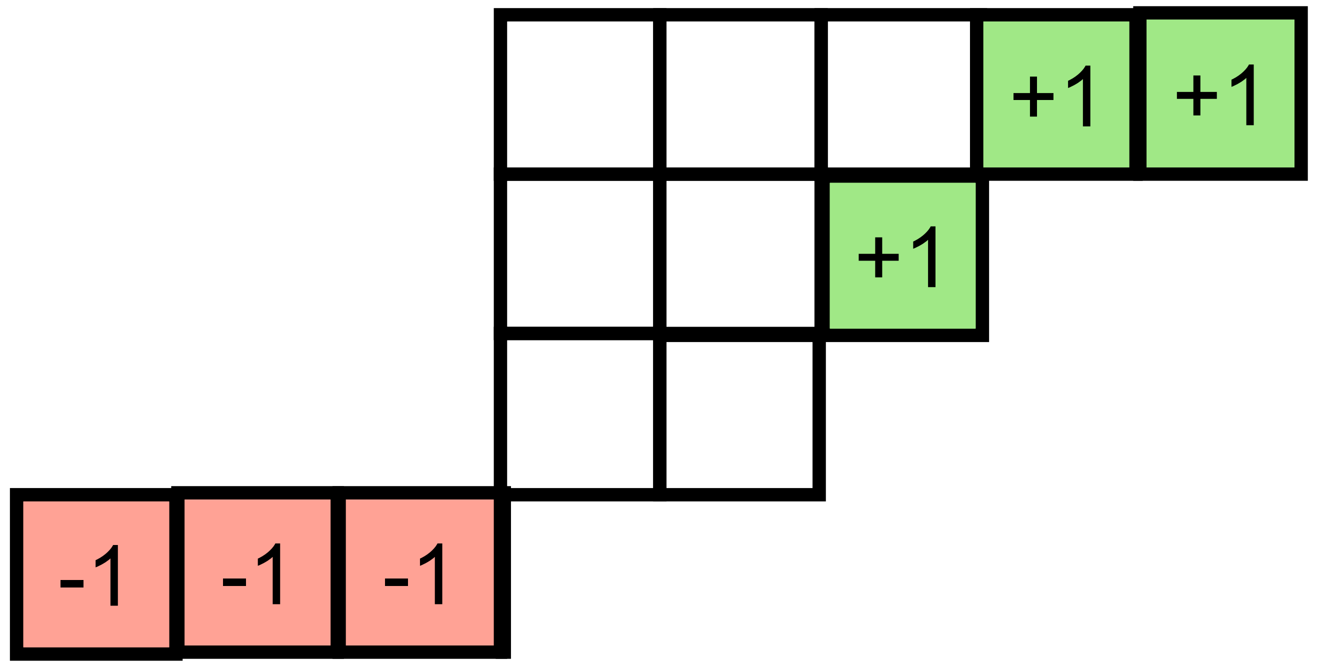


$\lambda = (3, 2, 2, 0)$

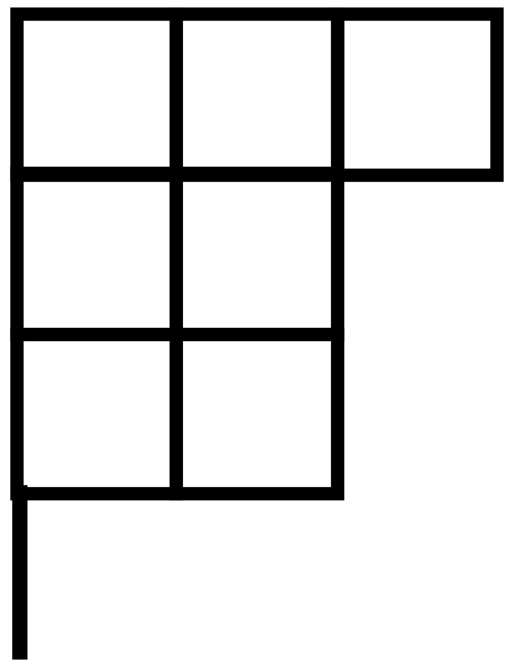
Rule. Each row "donates" one box to each row that is strictly longer.

Second row donates to first row.



What is $\lambda^{\uparrow}$?

Start from Young diagram λ.



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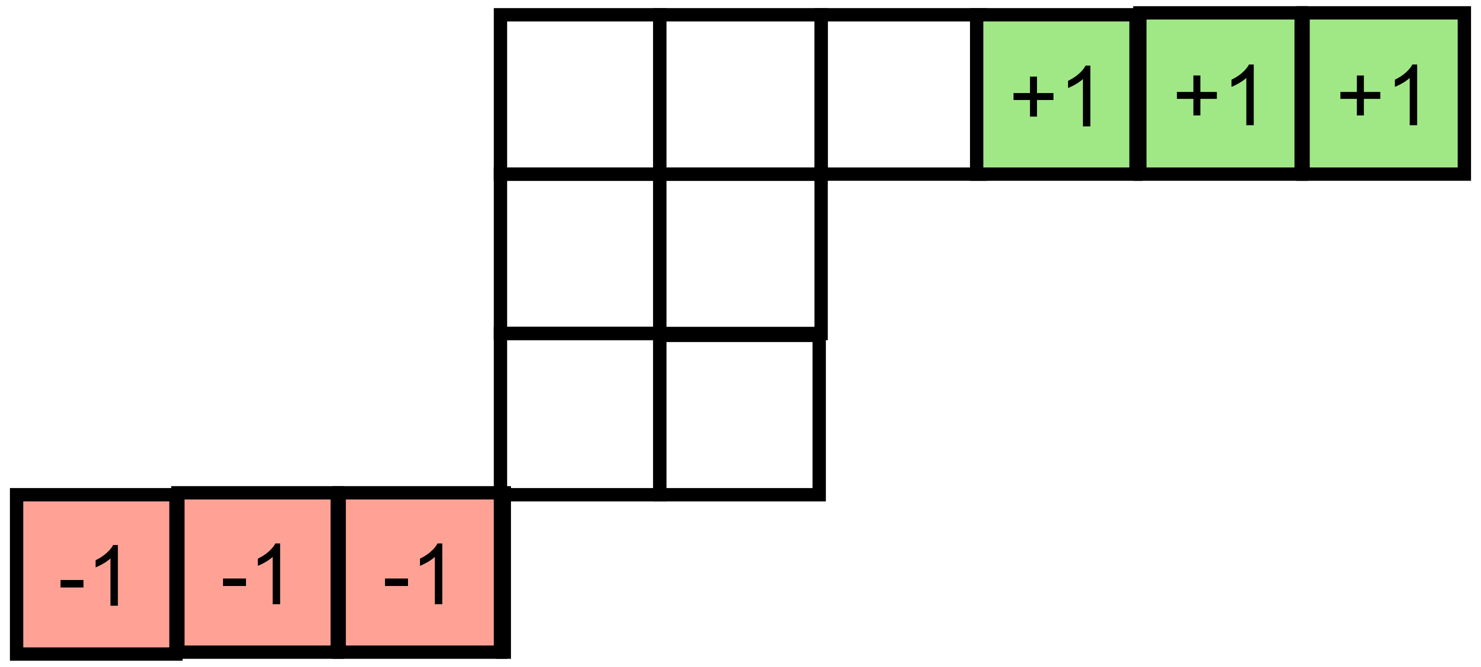

What is $\lambda^{\uparrow}$?

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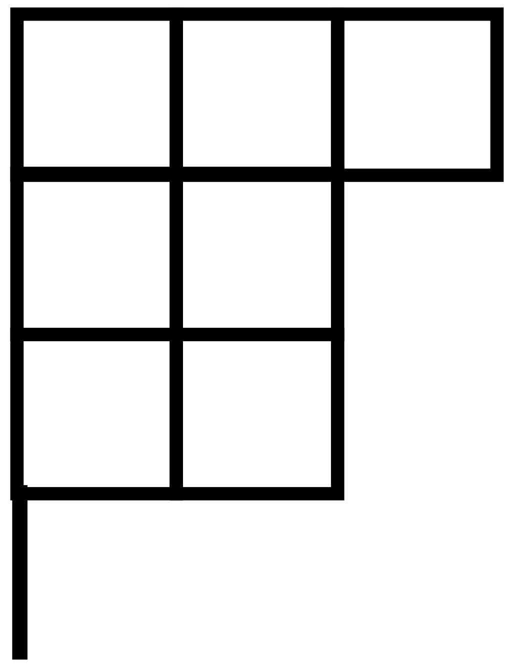
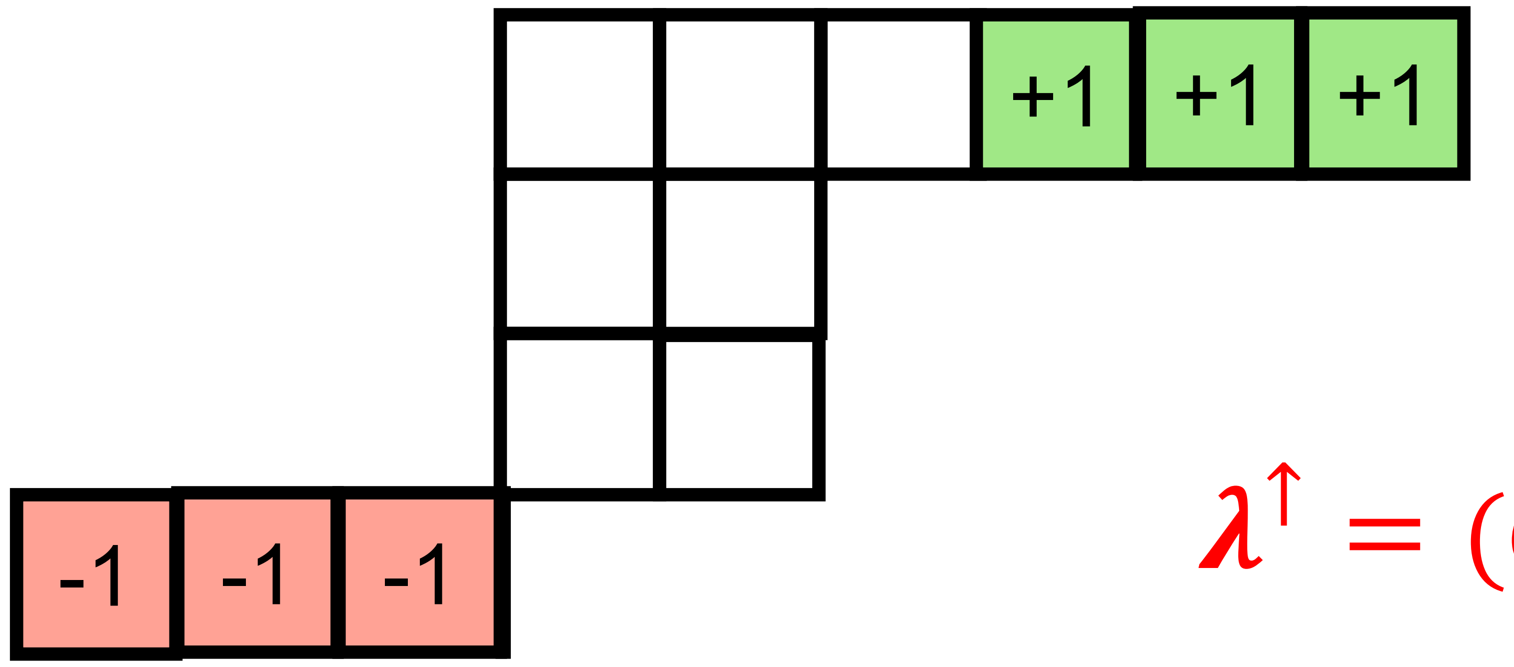


$\lambda = (3, 2, 2, 0)$

Rule. Each row "donates" one box to each row that is strictly longer.

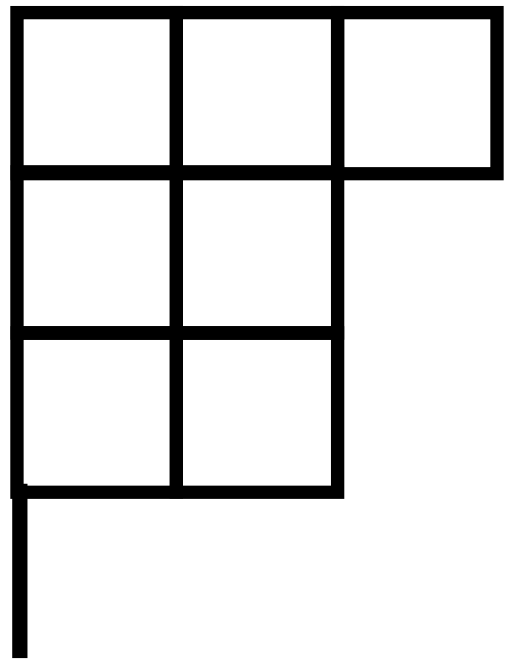
First row doesn't donate.



$\lambda^{\uparrow} = (6, 2, 2, -3)$

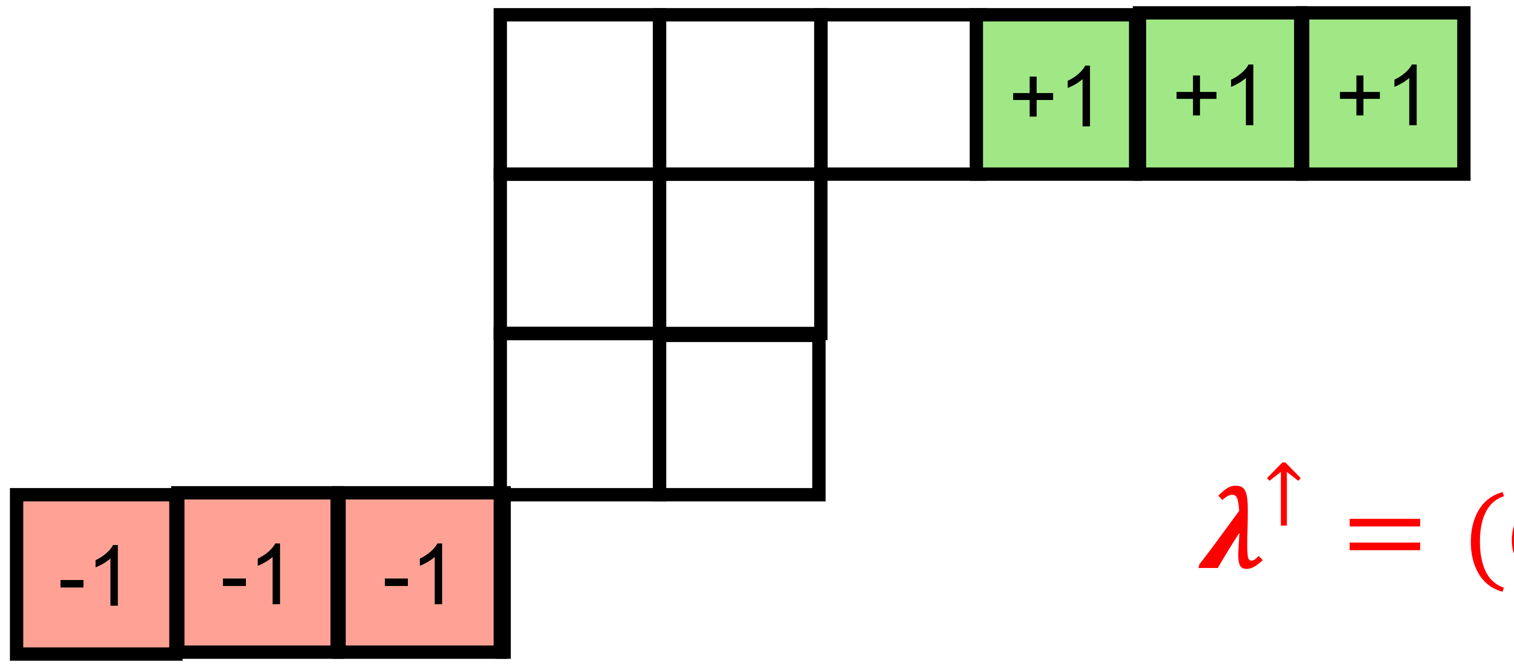
What is $\lambda^{\uparrow}$?

Start from Young diagram λ.



$\boldsymbol{\lambda = (3, 2, 2, 0)}$

Rule. Each row "donates" one box to each row that is strictly longer.



$\boldsymbol{\lambda^{\uparrow} = (6, 2, 2, -3)}$

Theorem. Matrix $\widehat{\boldsymbol{\rho}} = \widehat{\boldsymbol{U}} \cdot \mathrm{diag}\left(\boldsymbol{\lambda}^{\uparrow}/\boldsymbol{n}\right) \cdot \widehat{\boldsymbol{U}}^{-\mathbf{1}}$ is unbiased estimator of $\boldsymbol{\rho}$.

With the second moment we mentioned.

Conclusion

Conclusion

Unbiased Keyl's algorithm with a very clean modification.

Explicit and simple-to-use second moment formula.

Results. Sample-optimal tomography algorithms and new applications.

Follow-up [PSTW25]: New unbiased estimator via Random Purification Channel.

Same second moment $\Rightarrow$ Same applications!

Much simpler analysis.

But. Keyl's algorithm is provably optimal for state tomography [GMPW26].

Future directions.

1. Simpler analysis?
2. Higher moments of estimator?
3. Efficient implementation of the Keyl measurement?

Thank you!
Questions?